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Note

Jordan canonical form of Pascal-type matrices via sequences of binomial type

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Abstract

In this paper, we study the Jordan canonical form of the generalized Pascal functional matrix associated with a sequence of binomial type, and demonstrate that the transition matrix between the generalized Pascal functional matrix and its Jordan canonical form is the iteration matrix associated with the binomial sequence. In addition, some combinatorial identities are derived from the corresponding matrix factorization.

Keywords: Pascal matrix; Generalized Pascal functional matrix; Bell polynomial; Sequence of binomial type; Stirling number; Jordan canonical form; Iteration matrix

1. Introduction

The lower triangular Pascal matrix and its various generalizations were studied by many authors in the last two decades (see [1–5,12–14]). Let n be a positive integer, the $n \times n$ Pascal matrix P_n is defined with the binomial coefficients by [5]

$$P_n(i, j) = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For a parameter x, the Pascal functional matrices of order n is defined by

$$P_n[x](i, j) = \begin{cases} x^{i-j}\binom{i-1}{j-1}, & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

In [4,5], the authors discussed a few basic properties of the Pascal matrix, and they showed that the product formula for the Pascal functional matrices is $P_n[x]P_n[y] = P_n[x+y]$, and $P_n^m[x] = P_n[mx]$ for any integer m.

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In [13], Zhao and Wang defined the generalized Pascal functional matrix $\Phi_n[x]$ of order n as

$$\Phi_n[x](i,j) = \begin{cases} \varphi_{i-j}(x)\binom{i-1}{j-1}, & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{\varphi_n(x)\}$ is a sequence of binomial type, i.e., the sequence $\{\varphi_n(x)\}$ satisfying the following conditions:

(1) $\varphi_0(x) = 1$, $\varphi_1(x) = x$,
(2) for any positive integer n, $\varphi_n(x)$ is a polynomial of degree n with $\varphi_n(0) = 0$, and
(3) for all nonnegative integer n,

$$\varphi_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \varphi_k(x)\varphi_{n-k}(y). \tag{1}$$

The well-known Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$ are defined by the following generating functions [8, pp. 206 and 212]:

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{1}{k!}(\log(1+t))^k, \qquad \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{1}{k!}(e^t-1)^k.$$

Moreover, it is also known that the numbers $S(n,k)$ can be expressed as (see [8, p. 204, Theorem A]):

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n.$$

The Stirling matrix of the first kind s_n and the Stirling matrix of the second kind S_n are defined respectively by (see [6,7])

$$s_n(i,j) = \begin{cases} s(i-1,j-1), & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$S_n(i,j) = \begin{cases} S(i-1,j-1), & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For any parameter x, the generalized Stirling matrix of the first kind $s_n[x]$ and the generalized Stirling matrix of the second kind $S_n[x]$ are defined respectively by

$$s_n[x](i,j) = \begin{cases} x^{i-j}s(i-1,j-1), & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$S_n[x](i,j) = \begin{cases} x^{i-j}S(i-1,j-1), & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to check that $S_n s_n = I_n$ and $S_n[x]s_n[x] = I_n$, where I_n is the identity matrix of order n.

2. The Jordan canonical form of the generalized Pascal functional matrix

The Bell polynomials are the polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ in an infinite number of variables $x_1, x_2, \ldots$, defined by [8, p. 133]

$$\exp\left(u\sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n\geq 1}\frac{t^n}{n!}\left[\sum_{k=1}^{n} u^k B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})\right],$$

or, by the series expansion:

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}\frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots. \tag{2}$$

Based on (2), with every series $f(t) = \sum_{m=1}^{\infty} x_m t^m/m!$, we define the $n \times n$ iteration matrix $B_n(f)$ by (see [8, p. 145] and [10, p. 146])

$$B_n(f)(i,j) = \begin{cases} B_{i-1,j-1}, & \text{if } i \geq j \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $B_{i,j} = B_{i,j}(x_1, x_2, \ldots, x_{i-j+1})$. Particularly, if $f(t) = \sum_{m=1}^{\infty} \varphi_m(x) t^m/m!$, then the (i, j)th entry of $B_n(f)$ is $B_{i-1,j-1}(\varphi_1(x), \varphi_2(x), \ldots)$.

For example, from the generating functions, $s(n,k) = B_{n,k}(0!, -1!, 2!, \ldots)$ and $S(n,k) = B_{n,k}(1, 1, 1, \ldots)$. Thus, $B_n(f) = s_n$ when $f(t) = \log(1+t)$ and $B_n(f) = S_n$ when $f(t) = e^t - 1$.

The following recurrence relation can be found in [8, p. 136]:

Lemma 1. $kB_{n,k} = \sum_{j=k-1}^{n-1} \binom{n}{j} x_{n-j} B_{j,k-1}$.

With every infinite sequence $x_0 = 1, x_1, x_2, \cdots, x_n, \ldots$, we associate the $n \times n$ lower triangular matrix $Q_n = \left[x_{i-j}\binom{i-1}{j-1}\right]$, for each positive integer n. Let $D_n = diag(0!, 1!, 2!, \ldots, (n-1)!)$, $T_n = B_n D_n$, where B_n is the iteration matrix of the series $f(t) = \sum_{m=1}^{\infty} x_m t^m/m!$. Then it can be proved that the matrix Q_n is similar to a Jordan block matrix with transition matrix T_n.

Theorem 2. *Let J_n be the Jordan block of order n, that is,*

$$J_n(i,j) = \begin{cases} 1, & \text{if } i = j \text{ or } i = j+1, \\ 0, & \text{otherwise,} \end{cases}$$

then $Q_n = T_n J_n T_n^{-1}$.

Proof. It is sufficient to prove that $Q_n T_n = T_n J_n$. Applying Lemma 1, we get

$$\begin{aligned}
(Q_n T_n)(i,j) &= (Q_n B_n D_n)(i,j) \\
&= \sum_{m=j}^{i} x_{i-m} \binom{i-1}{m-1} B_{m-1,j-1}(j-1)! \\
&= \sum_{k=j-1}^{i-1} x_{i-k-1} \binom{i-1}{k} B_{k,j-1}(j-1)! \\
&= \sum_{k=j-1}^{i-2} x_{i-k-1} \binom{i-1}{k} B_{k,j-1}(j-1)! + x_0 B_{i-1,j-1}(j-1)! \\
&= B_{i-1,j}\, j! + B_{i-1,j-1}(j-1)! = (B_n D_n J_n)(i,j) = (T_n J_n)(i,j).
\end{aligned}$$

Hence, the result follows. □

The following result can be obtained from Theorem 2 immediately.

Theorem 3. *Let $\{\varphi_n(x)\}$ be a sequence of binomial type and $\Phi_n[x] = \left[\varphi_{i-j}(x)\binom{i-1}{j-1}\right]$ be the $n \times n$ generalized Pascal functional matrix associated with $\{\varphi_n(x)\}$. Let $g(t) = \sum_{n=1}^{\infty} \varphi_n(x) t^n/n!$, $B_n(g)$ be the iteration matrix for $g(t)$, and $T_n = B_n(g) D_n$. Then $\Phi_n[x] = T_n J_n T_n^{-1}$.*

Example 1. Let $\varphi_n(x) = x^n$, then the sequence is binomial obviously, $g(t) = \sum_{m=1}^{\infty} x^m t^m/m!$, and $B_{n,k}(x, x^2, x^3, \ldots) = x^n B_{n,k}(1, 1, 1, \ldots) = x^n S(n,k)$. Thus, $B_n(g) = E_n[x] S_n$ and $P_n[x] E_n[x] S_n D_n = E_n[x] S_n D_n J_n$, where $E_n[x] = \text{diag}(1, x, x^2, \ldots, x^{n-1})$. When $x \neq 0$,

$$P_n[x] = E_n[x] S_n D_n J_n D_n^{-1} s_n E_n^{-1}[x]. \tag{3}$$

From [9, p. 87, Exercise 37] and [11, Lemma 2], for a binomial sequence $\{\varphi_n(x)\}$ with exponential generating function $G(x,t) = \sum_{n=0}^{\infty} \varphi_n(x) t^n/n!$, there exists a power series $f(t) = \sum_{n=1}^{\infty} a_n t^n/n!$ with $a_1 = 1$ such that

$G(x, t) = \exp(xf(t))$, and

$$\varphi_n(x) = \sum_{k=0}^{n} x^k B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}). \tag{4}$$

We define $B_n(f)$ be the iteration matrix for $f(t)$. From [11, Theorem 2], we also have

$$B_{n,k}(\varphi_1(x), \varphi_2(x), \ldots, \varphi_{n-k+1}(x)) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\varphi_n(ix). \tag{5}$$

Combining (4) and (5) yields the following theorem.

Theorem 4. *For all integers $n, k \geq 0$,*

$$B_{n,k}(\varphi_1(x), \varphi_2(x), \ldots, \varphi_{n-k+1}(x)) = \sum_{j=k}^{n} B_{n,j}(a_1, a_2, \ldots, a_{n-j+1})S(j, k)x^j. \tag{6}$$

Proof. Expanding the right-hand side of (5) by (4) gives

$$\begin{aligned}
&\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{j=0}^{n}(ix)^j B_{n,j}(a_1, a_2, \cdots, a_{n-j+1}) \\
&\quad = \sum_{j=0}^{n} B_{n,j}(a_1, a_2, \ldots, a_{n-j+1})x^j \ \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^j \\
&\quad = \sum_{j=0}^{n} B_{n,j}(a_1, a_2, \ldots, a_{n-j+1})S(j, k)x^j.
\end{aligned}$$

This completes the proof. $\square$

Theorem 5. *Let $\{\varphi_n(x)\}$ be a sequence of binomial type, $\Phi_n[x] = \left[\varphi_{i-j}(x)\binom{i-1}{j-1}\right]$, then $B_n(g) = B_n(f)E_n[x]S_n$, and $\Phi_n[x] = B_n(f)P_n[x]B_n^{-1}(f)$.*

Proof. From Theorem 4, we have $B_n(g) = B_n(f)E_n[x]S_n$. By Theorem 3 and Eq. (3), we get

$$\begin{aligned}
\Phi_n[x] &= T_n J_n T_n^{-1} = B_n(g)D_n J_n(B_n(g)D_n)^{-1} \\
&= B_n(f)E_n[x]S_n D_n J_n(B_n(f)E_n[x]S_n D_n)^{-1} \\
&= B_n(f)E_n[x]S_n D_n J_n D_n^{-1} s_n E_n^{-1}[x]B_n^{-1}(f) = B_n(f)P_n[x]B_n^{-1}(f). \quad \square
\end{aligned}$$

From the matrix equation $\Phi_n[x]B_n(f) = B_n(f)P_n[x]$, we obtain the next result.

Theorem 6.

$$\sum_{k=j}^{i}\varphi_{i-k}(x)\binom{i-1}{k-1}B_{k-1,j-1}(a_1, a_2, \ldots, a_{k-j+1}) = \sum_{k=j}^{i}B_{i-1,k-1}(a_1, a_2, \ldots, a_{i-k+1})x^{k-j}\binom{k-1}{j-1}. \tag{7}$$

3. Applications

In this section, we obtain some identities for a few well-known binomial sequences by substituting them in Theorem 4 and Theorem 6 respectively.

Example 2. Let $\varphi_n(x) = x^{n|\lambda} = \begin{cases} x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), & \text{if } n \geq 1, \\ 1, & n = 0, \end{cases}$ then the sequence is binomial, where λ is an arbitrary parameter.

By computation, the exponential generating function for the sequence is

$$\sum_{n=0}^{\infty} \varphi_n(x)\frac{t^n}{n!} = \exp\left(x\frac{1}{\lambda}\log(1+\lambda t)\right),$$

then $f(t) = \frac{1}{\lambda}\log(1+\lambda t) = \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\lambda^{n-1}t^n/n!$, and we have

$$B_{n,k}(0!, -1!\lambda, 2!\lambda^2, \ldots, (-1)^{n-k}(n-k)!\lambda^{n-k}) = \lambda^{n-k}s(n,k).$$

It follows that $B_n(f) = s_n[\lambda]$ and for the $n \times n$ matrix $P_{n,\lambda}[x](i,j) = \left[x^{(i-j)|\lambda}\binom{i-1}{j-1}\right]$ we have $P_{n,\lambda}[x] = s_n[\lambda]P_n[x]S_n[\lambda]$.

Also, by Theorems 4 and 6, the following identities hold:

$$B_{n,k}(x, x(x-\lambda), x(x-\lambda)(x-2\lambda), \ldots) = \sum_{j=k}^{n}\lambda^{n-j}s(n,j)S(j,k)x^j,$$

$$\sum_{k=j}^{i} x^{(i-k)|\lambda}\binom{i-1}{k-1}\lambda^{k-j}s(k-1,j-1) = \sum_{k=j}^{i}\lambda^{i-k}x^{k-j}s(i-1,k-1)\binom{k-1}{j-1}.$$

Example 3. Let $\varphi_n(x) = x(x-na)^{n-1}$ be the Abel polynomials, then the sequence is binomial and the exponential generating function is

$$\sum_{n=0}^{\infty} x(x-an)^{n-1}\frac{t^n}{n!} = \exp\left(x\sum_{n=1}^{\infty}(-an)^{n-1}\frac{t^n}{n!}\right)$$

(see [9, pp. 87 and 132]). Using the equation [11, Eq.(27)]

$$B_{n,k}(1, -2a, (3a)^2, -(4a)^3, \ldots) = \binom{n-1}{k-1}(-na)^{n-k},$$

we have

$$B_{n,k}(x, x(x-2a), x(x-3a)^2, \ldots) = \sum_{j=k}^{n}\binom{n-1}{j-1}(-na)^{n-j}S(j,k)x^j,$$

$$\sum_{k=j}^{i} x(x-(i-k)a)^{i-k-1}\binom{i-1}{k-1}\binom{k-2}{j-2}(-(k-1)a)^{k-j} = \sum_{k=j}^{i}\binom{i-2}{k-2}(-(i-1)a)^{i-k}x^{k-j}\binom{k-1}{j-1}.$$

Example 4. Let $b_n(x) = \sum_{j=0}^{n} S(n,j)x^j$ be the exponential polynomials [9, p. 87]. The corresponding generating function is $\sum_{n=0}^{\infty} b_n(x)t^n/n! = e^{x(e^t-1)}$. Now $f(t) = e^t - 1 = \sum_{m=1}^{\infty} t^m/m!$, hence $B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) = B_{n,k}(1, 1, \ldots, 1) = S(n,k)$, and $B_n(f) = S_n$ is the Stirling matrix of the second kind.

Therefore, by Theorems 4 and 6, we get the following identities for the exponential polynomials

$$B_{n,k}(b_1(x), b_2(x), b_3(x), \ldots) = \sum_{j=k}^{n} S(n,j)S(j,k)x^j,$$

$$\sum_{k=j}^{i} b_{i-k}(x)\binom{i-1}{k-1}S(k-1,j-1) = \sum_{k=j}^{i} S(i-1,k-1)\binom{k-1}{j-1}x^{k-j}.$$

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